

# SUBINTEGRALITY, INVERTIBLE MODULES AND LAURENT POLYNOMIAL EXTENSIONS

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**ABSTRACT.** Let  $A \subseteq B$  be a commutative ring extension. Let  $\mathcal{I}(A, B)$  be the multiplicative group of invertible  $A$ -submodules of  $B$ . In this article, we extend Sadhu and Singh result by finding a necessary and sufficient condition on  $A \subseteq B$ , so that the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism. We also discuss some properties of the cokernel of the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  in general case.

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## INTRODUCTION

In [3], Roberts and Singh have introduced the group  $\mathcal{I}(A, B)$  to generalize a result of Dayton. The relation between the group  $\mathcal{I}(A, B)$  and subintegral extensions has been investigated by Reid, Roberts and Singh in a series of papers. Recently in [4], Sadhu and Singh have proved that  $A$  is subintegrally closed in  $B$  if and only if the canonical map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X], B[X])$  is an isomorphism. It is easy to see that  $\mathcal{I}(A[X], B[X]) = \mathcal{I}(A, B) \oplus N\mathcal{I}(A, B)$ . So the result of [4], just mentioned amounts to saying that  $N\mathcal{I}(A, B) = 0$  if and only if  $A$  is subintegrally closed in  $B$ .

The primary goal of this paper is to extend Sadhu and Singh result of [4] just mentioned above by finding a necessary and sufficient condition on  $A \subseteq B$ , so that the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism. It is easy to see that the map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is always injective. The secondary goal is to investigate the cokernel of the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  in general case. This cokernel will be denoted by  $M\mathcal{I}(A, B)$ .

In Section 1, we mainly give basic definitions and notations.

In Section 2, we discuss conditions on  $A \subseteq B$  under which the map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism. We show that for an integral, birational one dimensional domain extension  $A \subseteq B$ , the map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism if and only if  $A$  is subintegrally closed in  $B$  and  $A \subseteq B$  is anodal. We give an example to show that the map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  need not be

an isomorphism for a 2 dimensional extension even if  $A$  is subintegrally closed in  $B$  and  $A \subseteq B$  is anodal.

In Section 3, we discuss the surjectivity of the natural map  $\varphi(A, C, B) : \mathcal{I}(A, B) \rightarrow \mathcal{I}(C, B)$  is given by  $\varphi(A, C, B)(I) = IC$  for any ring extensions  $A \subseteq C \subseteq B$ . We show that the map  $\varphi(A, C, B)$  is surjective if  $C$  is subintegral over  $A$ . We show further that if  $C$  subintegral over  $A$ , then the sequence

$$1 \rightarrow MI(A, C) \rightarrow MI(A, B) \rightarrow MI(C, B) \rightarrow 1$$

is exact. We conclude this section by discussing some properties of the group  $MI(A, B)$ .

### 1. BASIC DEFINITIONS AND NOTATIONS

All of the rings we consider are commutative with 1, and all ring homomorphisms are unitary. Let  $X, T$  be indeterminates.

An **elementary subintegral** extension is an extension of the form  $A \subseteq B$  with  $B = A[b]$  for some  $b \in B$  such that  $b^2, b^3 \in A$ . An extension  $A \subseteq B$  is **subintegral** if it is a filtered union of elementary subintegral extensions; that is, for each  $b \in B$  there is a finite sequence  $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_r \subseteq B$  of ring extensions such that  $b \in C_r$  and  $C_{i-1} \subseteq C_i$  is elementary subintegral for each  $i$ ,  $1 \leq i \leq r$ . We say that  $A$  is **subintegrally closed** in  $B$  if whenever  $b \in B$  and  $b^2, b^3 \in A$  then  $b \in A$ . The ring  $A$  is **seminormal** if the following condition holds:  $b, c \in A$  and  $b^3 = c^2$  imply that there exists  $a \in A$  with  $b = a^2$  and  $c = a^3$ . A seminormal ring is necessarily reduced and is subintegrally closed in every reduced overring. It is easily seen that if  $A$  is subintegrally closed in  $B$  with  $B$  seminormal then  $A$  is seminormal. For details see [6, 7].

For a ring  $A$  we denote by :

$U(A)$ : The groups of units of  $A$ .

$H^0(A) = H^0(\text{Spec} A, \mathbb{Z})$ : The group of continuous maps from  $\text{Spec}(A)$  to  $\mathbb{Z}$ .

$\text{Pic} A$ : The Picard group of  $A$ .

$KU(A)$ : Cokernel of the natural map  $U(A) \rightarrow U(A[X])$ .

$MU(A)$  : Cokernel of the natural map  $U(A) \rightarrow U(A[X, X^{-1}])$ .

$NU(A)$ : Kernel of the map  $U(A[X]) \rightarrow U(A)$ .

$\text{KPic} A$  : Cokernel of the natural map  $\text{Pic} A \rightarrow \text{Pic} A[X]$ .

$\text{MPic} A$  : Cokernel of the natural map  $\text{Pic} A \rightarrow \text{Pic} A[X, X^{-1}]$ .

$\text{NPic} A$ : Kernel of the map  $\text{Pic} A[X] \rightarrow \text{Pic} A$ .

$\text{LPic} A$ : Cokernel of the map  $\text{Pic} A[X] \times \text{Pic} A[X^{-1}] \xrightarrow{\text{add}} \text{Pic} A[X, X^{-1}]$ .

Let  $A \subseteq B$  be a ring extension. Then we denote by

$\mathcal{I}(A, B)$ : The group of invertible  $A$ - submodules of  $B$ .

It is easily seen that  $\mathcal{I}$  is a functor from extensions of rings to abelian groups. Some properties of  $\mathcal{I}(A, B)$  can be found in [3, Section 2].

$K\mathcal{I}(A, B)$ : Cokernel of the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X], B[X])$ .

$M\mathcal{I}(A, B)$ : Cokernel of the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ .

$N\mathcal{I}(A, B)$ : Kernel of the map  $\mathcal{I}(A[X], B[X]) \rightarrow \mathcal{I}(A, B)$  (Here the map is induced by the  $B$ -algebra homomorphism  $B[X] \rightarrow B$  given by  $X \mapsto 0$ ).

Recall from [3, Section 2] that for any commutative ring extension  $A \subseteq B$ , we have the exact sequence

$$1 \rightarrow U(A) \rightarrow U(B) \rightarrow \mathcal{I}(A, B) \rightarrow \text{Pic } A \rightarrow \text{Pic } B.$$

Applying  $M, K$  we obtain the chain complexes:

$$(1.0) \quad 1 \rightarrow MU(A) \rightarrow MU(B) \rightarrow M\mathcal{I}(A, B) \xrightarrow{\eta} MPic A \xrightarrow{\varphi} MPic B.$$

and

$$(1.1) \quad 1 \rightarrow KU(A) \rightarrow KU(B) \rightarrow K\mathcal{I}(A, B) \xrightarrow{\alpha} KPic A \xrightarrow{\beta} KPic B.$$

## 2. THE MAP $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$

In this section we examine some conditions on  $A \subseteq B$  under which the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism (i.e  $M\mathcal{I}(A, B) = 0$ ). For this we consider the notions of quasinormal and anodal extensions.

Let  $A \subseteq B$  be a ring extension. We say that  $A$  is **quasinormal** in  $B$  if the natural map  $MPic A \rightarrow MPic B$  is injective. For properties see [2].

The following result is due to Sadhu and Singh [4] which we use frequently throughout this paper:

**Lemma 2.1.** *Let  $A \subseteq B$  be a ring extension. Then  $A$  is subintegrally closed in  $B$  if and only if the canonical map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X], B[X])$  is an isomorphism.*

*Proof.* See Theorem 1.5 of [4]. □

One can restate the above result in the following way:  $A$  is subintegrally closed in  $B$  if and only if  $K\mathcal{I}(A, B) = 0$  if and only if  $N\mathcal{I}(A, B) = 0$ .

The following result is due to Weibel [8]

**Lemma 2.2.** *There is a natural decomposition*

$$\text{Pic } A[X, X^{-1}] \cong \text{Pic } A \oplus N\text{Pic } A \oplus N\text{Pic } A \oplus L\text{Pic } A$$

*for any commutative ring  $A$ .*

*Proof.* See Theorem 5.2 of [8]. □

**Remark 2.3.** By Swan Theorem [6],  $\text{NPic } A = 0$  if and only if  $A_{\text{red}}$  is seminormal. So for a seminormal ring  $A$ ,  $\text{LPic } A \cong \text{MPic } A$ .

**Lemma 2.4.** There is a natural decomposition

$$U(A[X, X^{-1}]) \cong U(A) \oplus NU(A) \oplus NU(A) \oplus H^0(A)$$

for any commutative ring  $A$ .

*Proof.* See Exercise 3.17 of [9] in page 30.  $\square$

**Remark 2.5.** So for a reduced ring  $A$ ,  $H^0(A) \cong MU(A)$ .

**Lemma 2.6.** The natural map  $\phi : \mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ , given by  $I \rightarrow IA[X, X^{-1}]$  is injective.

*Proof.* Let  $I = (b_1, b_2, \dots, b_r)A \in \text{Ker } \phi$ , where  $b_i \in B$ . Then  $IA[X, X^{-1}] = A[X, X^{-1}]$ . This implies that  $b_i \in A[X, X^{-1}] \cap B = A$ , for all  $i$ . So  $I \subseteq A$ . Similarly  $I^{-1} \subseteq A$ . Hence  $I = A$ .  $\square$

**Lemma 2.7.** The sequence (1.0)[resp. (1.1)] is exact, except possibly at the place  $\text{MPic } A$ [resp.  $\text{KPic } A$ ]. It is exact there too if the map  $\text{Pic } A \rightarrow \text{Pic } B$  is surjective.

*Proof.* We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & U(A) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \text{Pic } A & \longrightarrow & \text{Pic } B \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & U(A[X, X^{-1}]) & \longrightarrow & U(B[X, X^{-1}]) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & \text{Pic } A[X, X^{-1}] & \longrightarrow & \text{Pic } B[X, X^{-1}] \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & MU(A) & \longrightarrow & MU(B) & \longrightarrow & M\mathcal{I}(A, B) & \longrightarrow & \text{MPic } A & \longrightarrow & \text{MPic } B \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1 & & 0 & & 0
 \end{array}$$

where first two rows are exact and each column is exact. Now result follows by chasing this diagram.  $\square$

**Lemma 2.8.** Let  $A \subseteq B$  be a ring extension. The map  $\text{Pic } A \rightarrow \text{Pic } B$  is surjective if any one of the following conditions holds

- (1)  $A \subseteq B$  is subintegral.
- (2)  $A \subseteq B$  is a birational integral extension of domains, with  $\dim A = 1$ .

*Proof.* (1) See Proposition 7 of [1].

(2) Let  $K$  be the quotient field of  $A$  and  $B$ . We have the commutative diagram

$$\begin{array}{ccccc} \mathcal{I}(A, K) & \longrightarrow & \text{Pic } A & \longrightarrow & 0 \\ \theta(A, B, K) \downarrow & & \varphi \downarrow & & \\ \mathcal{I}(B, K) & \longrightarrow & \text{Pic } B & \longrightarrow & 0 \end{array}$$

where  $\theta(A, B, K)$  is surjective by Proposition 2.3 of [4]. Hence  $\varphi$  is surjective.  $\square$

**Lemma 2.9.** *Let  $A \subseteq B$  be a ring extension with  $B$  a domain. Then*

(1) *If  $A$  is quasinormal in  $B$  then  $M\mathcal{I}(A, B) = 0$ .*

(2) *Suppose the extension  $A \subseteq B$  is integral and birational with  $\dim A \leq 1$  and  $M\mathcal{I}(A, B) = 0$ . Then  $A$  is quasinormal in  $B$ .*

*Proof.* (1) Since  $A$  and  $B$  are domains,  $MU(A) = MU(B) \cong \mathbb{Z}$ . By (1.0),  $\text{Im } \eta \subseteq \ker \varphi$ . As  $A$  is quasinormal in  $B$ ,  $\ker \varphi = 0$ . This implies that  $\text{Im } \eta = 0$ . We get  $M\mathcal{I}(A, B) = 0$ .

(2) By Lemma 2.8(2) and Lemma 2.7, the sequence (1.0) is exact at  $MPic A$  also. Since  $M\mathcal{I}(A, B) = 0$ , we get the result.  $\square$

**Lemma 2.10.** (cf. [2], Lemma 1.4.) *Let  $A \subseteq B$  be a ring extension with  $B$  reduced and  $A$  quasinormal in  $B$ . Then  $A$  is subintegrally closed in  $B$ .*

*Proof.* We have not assumed  $B$  to be a domain. By Lemma 2.1, it is enough to show that  $K\mathcal{I}(A, B) = 0$ . We have the sequence

$$1 \rightarrow KU(A) \rightarrow KU(B) \rightarrow K\mathcal{I}(A, B) \xrightarrow{\alpha} KPic A \xrightarrow{\beta} KPic B.$$

which is exact except possibly at the place  $KPic A$ . Since  $A$  and  $B$  are reduced,  $KU(A) = 0$  and  $KU(B) = 0$ . In the proof of Lemma 1.4 [2], it is shown that the map  $KPic A \rightarrow KPic B$  is injective i.e  $\ker \beta = 0$ . We have  $\text{im } \alpha \subseteq \ker \beta$ . Hence  $K\mathcal{I}(A, B) = 0$ .  $\square$

Note that in the above lemma we cannot drop the condition that  $B$  is reduced. For example, consider the extension  $A = K \subsetneq B = K[b]$  with  $b^2 = 0$ , where  $K$  is any field. Since  $MPic K = 0$ , clearly  $A$  is quasinormal in  $B$ . But  $A$  is not subintegrally closed in  $B$ , because  $b^2 = b^3 = 0 \in K$ ,  $b \notin K$ .

An inclusion  $A \subseteq B$  of rings is called **anodal** or **an anodal extension**, if every  $b \in B$  such that  $(b^2 - b) \in A$  and  $(b^3 - b^2) \in A$  belongs to  $A$ .

**Lemma 2.11.** *Let  $A \subseteq C \subseteq B$  be extensions of rings. Then*

(1) *If  $A$  is anodal in  $B$ , then so is  $A$  in  $C$ .*

(2) If  $A$  is anodal in  $C$  and  $C$  is anodal in  $B$ , then so is  $A$  in  $B$ .

*Proof.* Clear from the definition.  $\square$

**Proposition 2.12.** *Let  $A \subseteq B$  be a ring extension. If  $A \subseteq B$  is subintegral, then it is anodal.*

*Proof.* Assume first that  $A \subseteq B$  is an elementary subintegral extension i.e  $A \subseteq B = A[b]$  such that  $b^2, b^3 \in A$ . Let  $f \in B$  such that  $f^2 - f, f^3 - f^2 \in A$ . We have to show that  $f \in A$ . Clearly  $f$  is of the form  $a + \lambda b$  where  $a, \lambda \in A$ . So it is enough to show that  $\lambda b \in A$ . Since  $\lambda b(2a - 1), \lambda b(3a^2 - 1) \in A$ ,  $\lambda b = \lambda b \cdot 1 = \lambda b[(6a + 3)(2a - 1) - 4(3a^2 - 1)] \in A$ . Hence  $f \in A$ .

In the general case, for  $f \in B$  there exists a finite sequence  $A = C_0 \subseteq C_1 \subseteq \dots \subseteq C_r \subseteq B$  of extensions such that  $C_i \subseteq C_{i+1}$  is an elementary subintegral extension for each  $i, 0 \leq i \leq r - 1$  and  $f \in C_r$ . So by the above argument  $C_i \subseteq C_{i+1}$  is anodal for each  $i$ . Now the result follows from Lemma 2.11(2).  $\square$

**Lemma 2.13.** (1) *The diagram*

$$\begin{array}{ccc} \mathcal{I}(A, B) & \xrightarrow{\theta_1} & \mathcal{I}(A[X], B[X]) \\ & \searrow \theta & \downarrow \theta_2 \\ & & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \end{array}$$

*is commutative.*

(2) *the maps  $\theta, \theta_1$  and  $\theta_2$  are injective.*

(3)  *$\theta$  is an isomorphism if and only if  $\theta_1$  and  $\theta_2$  are isomorphisms.*

(4) *If  $\theta$  is an isomorphism i.e  $MI(A, B) = 0$  then  $A$  is subintegrally closed in  $B$ .*

*Proof.* (1) Since the maps are natural, the diagram is commutative.

(2)  $\theta$  is injective by Lemma 2.6. The injectivity of  $\theta_1$  and  $\theta_2$  follows by similar argument as Lemma 2.6.

(3) If  $\theta_1$  and  $\theta_2$  are isomorphisms then clearly  $\theta$  is an isomorphism. Conversely, suppose  $\theta$  is an isomorphism. Then by simple diagram chasing we get that  $\theta_1$  and  $\theta_2$  are isomorphisms.

(4) If  $\theta$  is an isomorphism then  $\theta_1$  is an isomorphism. Hence by Lemma 2.1,  $A$  is subintegrally closed in  $B$ .  $\square$

**Lemma 2.14.** *Let  $\mathfrak{a}$  be a  $B$ -ideal contained in  $A$ . Then the homomorphism  $MI(A, B) \rightarrow MI(A/\mathfrak{a}, B/\mathfrak{a})$  is an isomorphism.*

*Proof.* Clearly,  $\mathfrak{a}[X, X^{-1}]$  is a  $B[X, X^{-1}]$ -ideal contained in  $A[X, X^{-1}]$ . We have  $\mathcal{I}(A, B) \cong \mathcal{I}(A/\mathfrak{a}, B/\mathfrak{a})$  and  $\mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \cong \mathcal{I}(A/\mathfrak{a}[X, X^{-1}], B/\mathfrak{a}[X, X^{-1}])$  by Proposition 2.6 of [3]. Now by chasing a suitable diagram we get the result.  $\square$

**Theorem 2.15.** (1) *Let  $A \subseteq B$  be an integral, birational extension of domains. Suppose  $M\mathcal{I}(A, B) = 0$ . Then  $A \subseteq B$  is anodal.*

(2) *Let  $A \subseteq B$  be an integral, birational extension of one dimensional domains with  $A \subseteq B$  anodal and  $A$  subintegrally closed in  $B$ . Then  $M\mathcal{I}(A, B) = 0$ .*

*Proof.* (1) By Lemma 2.13(4),  $A$  is subintegrally closed in  $B$ . Then by Lemma 1.10 of [2], it is enough to show that for every intermediate ring  $C$  between  $A$  and  $B$  such that  $C$  is a finite  $A$ -module, the map  $MPic A \rightarrow MPic A/\mathfrak{c} \times MPic C$  is injective, where  $\mathfrak{c}$  is the conductor of  $C$  in  $A$ . We first claim that the map  $\phi : M\mathcal{I}(A, C) \rightarrow M\mathcal{I}(A, B)$  is injective, where  $C$  is any intermediate ring between  $A$  and  $B$ .

We have the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{I}(A, C) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, C) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \phi \\ 1 & \longrightarrow & \mathcal{I}(A, B) & \xrightarrow{\beta} & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, B) \longrightarrow 1 \end{array}$$

where the first two vertical arrows are natural inclusions (because any invertible  $A$ -submodule of  $C$  is also an invertible  $A$ -submodule of  $B$ ).

Let  $\bar{J} \in \ker \phi$ , where  $J \in \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}])$ . Then  $J \in im \beta$  and there exist  $J_1 \in \mathcal{I}(A, B)$  such that  $J_1 A[X, X^{-1}] = J$ . Let  $J_1 = (b_1, b_2, \dots, b_r)A$  and  $J = (f_1, f_2, \dots, f_s)A[X, X^{-1}]$  where  $b_i \in B$  and  $f_i \in C[X, X^{-1}]$ . Then clearly  $b_i \in B \cap C[X, X^{-1}] = C$  for all  $i$ . So  $J_1 \subseteq C$ . Also  $J_1^{-1} \subseteq C$ . This implies that  $J_1 \in \mathcal{I}(A, C)$ . So  $\bar{J} = 0$ . This proves the claim.

Since  $M\mathcal{I}(A, B) = 0$ ,  $M\mathcal{I}(A, C) = 0$ . By Lemma 2.14,  $M\mathcal{I}(A/\mathfrak{c}, C/\mathfrak{c}) = 0$ , where  $\mathfrak{c}$  is the conductor of  $C$  in  $A$ . By (1.0), we have  $MU(A) \cong MU(C)$  and  $MU(A/\mathfrak{c}) \cong MU(C/\mathfrak{c})$ . Now the result follows from the following exact sequence which we obtain by applying  $M$  to the unit-Pic sequence (see Theorem 3.10, [9]),

$$MU(A) \rightarrow MU(A/\mathfrak{c}) \times MU(C) \rightarrow MU(C/\mathfrak{c}) \rightarrow MPic A \rightarrow MPic A/\mathfrak{c} \times MPic C$$

(2) By Theorem 1.13 of [2],  $A$  is quasinormal in  $B$ . Now the result follows from Lemma 2.9(1).  $\square$

**Corollary 2.16.** *Let  $A \subseteq B$  be an integral, birational extension of one dimensional domains. Then the following are equivalent:*

(1)  *$A$  is quasinormal in  $B$ .*

(2)  $A \subseteq B$  anodal and  $A$  is subintegrally closed in  $B$ .

(3)  $M\mathcal{I}(A, B) = 0$ .

*Proof.* (1)  $\Leftrightarrow$  (3) This is Lemma 2.9.

(2) $\Rightarrow$  (3) This is Theorem 2.15(2).

(3) $\Rightarrow$  (2) This follows from Lemma 2.13(4) and Theorem 2.15(1).  $\square$

The statement of Theorem 2.15(2) need not be true for dimension greater than 1. This is seen by considering Example 3.5 of [8]. In that example  $A$  is a 2- dimensional noetherian domain whose integral closure is  $B = K[X, Y]$ , where  $K$  is a field. So  $A \subseteq B$  is an integral, birational extension. By Proposition 3.5.2 of [8],  $A \subseteq B$  is anodal and  $A$  is subintegrally closed in  $B$ . Since  $B$  is a UFD,  $\text{Pic } B = \text{Pic } B[T, T^{-1}] = 0$  and we have the exact sequence

$$1 \rightarrow MU(A) \rightarrow MU(B) \rightarrow M\mathcal{I}(A, B) \rightarrow MPic A \rightarrow 0.$$

As  $A, B$  are domains,  $MU(A) = MU(B) \cong \mathbb{Z}$ . So  $M\mathcal{I}(A, B) \cong MPic A$ . By Remark 2.3,  $LPic A \cong MPic A$ . Hence by Proposition 3.5.2 of [8],  $M\mathcal{I}(A, B) \neq 0$ .

### 3. SOME OBSERVATIONS ON $M\mathcal{I}(A, B)$

Recall from [5, Section 3] that for any extensions  $A \subseteq C \subseteq B$  of rings, we have the exact sequence

$$1 \rightarrow \mathcal{I}(A, C) \rightarrow \mathcal{I}(A, B) \xrightarrow{\varphi(A, C, B)} \mathcal{I}(C, B)$$

where the map  $\varphi(A, C, B)$  is given by  $\varphi(A, C, B)(I) = IC$ .

Now it is natural to ask under what conditions on  $A \subseteq B$  the map  $\varphi(A, C, B)$  is surjective. In [5], Singh has proved that if  $B$  is subintegral over  $A$  then the map  $\varphi(A, C, B)$  is surjective. In the next Proposition we generalize Singh's result as follows:

**Proposition 3.1.** *For all rings  $C$  between  $A$  and  $B$  such that  $C$  is subintegral over  $A$ , the map  $\varphi(A, C, B)$  is surjective.*

*Proof.* We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(A) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \text{Pic } A & \longrightarrow & \text{Pic } B \\ & & \downarrow & & \downarrow = & & \downarrow \varphi(A, C, B) & & \downarrow \theta & & \downarrow = \\ 1 & \longrightarrow & U(C) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(C, B) & \longrightarrow & \text{Pic } C & \longrightarrow & \text{Pic } B \end{array}$$

Since  $\theta$  is surjective by Lemma 2.8(1), the result follows by chasing the diagram.  $\square$



Recall that a local ring  $A$  is **hensel** if every finite  $A$ -algebra  $B$  is a direct product of local rings.

The following result gives another case where the map  $\varphi(A, C, B)$  is surjective.

**Proposition 3.2.** *Let  $A \subseteq B$  be an integral extension with  $A$  hensel local. Then for all rings  $C$  with  $A \subseteq C \subseteq B$  the map  $\varphi(A, C, B)$  is surjective.*

*Proof.* By Lemma 2.2 of [4], it is enough to show that  $\varphi(A, D, B)$  is surjective for every subring  $D$  of  $C$  containing  $A$  such that  $D$  is finitely generated as an  $A$ -algebra. Let such a ring  $D$  be given. Since  $D$  is integral over  $A$ ,  $D$  is a finite  $A$ -algebra. As  $A$  is hensel,  $D$  is a finite direct product of local rings. Then  $\text{Pic } A$  and  $\text{Pic } D$  are both trivial. This implies that  $\mathcal{I}(A, B) = U(B)/U(A)$ ,  $\mathcal{I}(D, B) = U(B)/U(D)$  and clearly  $\varphi(A, D, B)$  is surjective.  $\square$

**Proposition 3.3.** *Let  $A \subseteq C \subseteq B$  be extensions of rings with  $A \subseteq C$  subintegral. Then the sequence*

$$1 \rightarrow MI(A, C) \rightarrow MI(A, B) \rightarrow MI(C, B) \rightarrow 1$$

*is exact.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathcal{I}(A, C) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}]) & \longrightarrow & MI(A, C) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & MI(A, B) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathcal{I}(C, B) & \longrightarrow & \mathcal{I}(C[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & MI(C, B) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

where the rows are clearly exact. Since  $A \subseteq C$  is subintegral, so is  $A[X, X^{-1}] \subseteq C[X, X^{-1}]$ . Therefore by Proposition 3.1, the first two columns are exact. Hence exactness of the last column follows by chasing the diagram.  $\square$

**Corollary 3.4.** *Let  $A \subseteq B$  be a ring extension and let  ${}^+A$  denote the subintegral closure of  $A$  in  $B$ . Then the sequence*

$$1 \rightarrow MI(A, {}^+A) \rightarrow MI(A, B) \rightarrow MI({}^+A, B) \rightarrow 1$$

*is exact.*

*Proof.* We have  $A \subseteq {}^+A \subseteq B$  where  $A \subseteq {}^+A$  is subintegral and  ${}^+A$  is subintegrally closed in  $B$ . By Proposition 3.1, we have the exact sequence

$$1 \rightarrow \mathcal{I}(A, {}^+A) \rightarrow \mathcal{I}(A, B) \xrightarrow{\varphi(A, {}^+A, B)} \mathcal{I}({}^+A, B) \rightarrow 1$$

Applying  $M$  we also get the following exact sequence by Proposition 3.3,

$$1 \rightarrow M\mathcal{I}(A, {}^+A) \rightarrow M\mathcal{I}(A, B) \rightarrow M\mathcal{I}({}^+A, B) \rightarrow 1$$

Hence the proof. □

**Proposition 3.5.** *Let  $A \subseteq B$  be a ring extension. Assume that  $A$  is subintegrally closed in  $B$ . Then*

- (1)  $M\mathcal{I}(A, B) \cong M\mathcal{I}(A[T], B[T])$ .
- (2)  $M\mathcal{I}(A, B)$  is a torsion free abelian group if  $B$  is a seminormal ring.
- (3)  $M\mathcal{I}(A, B)$  is a free abelian group if  $B$  is a seminormal ring and  $A$  is hensel local.
- (4)  $M\mathcal{I}(A, B) = 0$  if  $B$  is a seminormal domain and  $A$  is hensel local.

*Proof.* (1) Since  $A$  is subintegrally closed in  $B$ ,  $A[X]$  is subintegrally closed in  $B[X]$  by Corollary 1.6 of [4]. Therefore  $A[X, X^{-1}]$  is subintegrally closed in  $B[X, X^{-1}]$ . We have the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, B) \longrightarrow 1 \\ & & \downarrow \beta & & \downarrow \theta & & \downarrow \\ 1 & \longrightarrow & \mathcal{I}(A[T], B[T]) & \longrightarrow & \mathcal{I}(A[T][X, X^{-1}], B[T][X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A[T], B[T]) \longrightarrow 1 \end{array}$$

where  $\beta$  and  $\theta$  are isomorphisms by Lemma 2.1. Hence we get the result.

(2) As  $A$  is subintegrally closed in  $B$  and  $B$  is a seminormal ring,  $A$  is seminormal. Then by Remark 2.3,  $LPic A \cong MPic A$ . Since seminormal ring is reduced,  $MU(A) = H^0(A)$  and  $MU(B) = H^0(B)$  by Remark 2.5. Now, we have the exact sequence

$$1 \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow M\mathcal{I}(A, B) \rightarrow MPic A$$

where  $H^0(A)$  and  $H^0(B)$  are always free abelian groups by Construction 1.2.1 of [8] and by Corollary 2.3.1 of [8],  $MPic A$  is a torsion free abelian group. Let  $T = \text{Coker}[H^0(A) \rightarrow H^0(B)]$ . Then

$$1 \rightarrow T \rightarrow M\mathcal{I}(A, B) \rightarrow MPic A$$

is exact and  $T$  is a free abelian group by Proposition 1.3 of [8]. Therefore  $M\mathcal{I}(A, B)$  is a torsion free abelian group.

(3) By Theorem 2.5 of [8],  $LPic A = 0$ . Since  $A$  is seminormal,  $MPic A = 0$ . Then we have the exact sequence

$$1 \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow M\mathcal{I}(A, B) \rightarrow 1$$

and  $M\mathcal{I}(A, B) = \text{Coker}[H^0(A) \rightarrow H^0(B)]$  is a free abelian group by Proposition 1.3 of [8].

(4) Since  $B$  is a domain,  $H^0(A) = H^0(B) \cong \mathbb{Z}$ . So  $M\mathcal{I}(A, B) = 0$ . □

**Lemma 3.6.** *Let  $A \subseteq B$  be a subintegral extension. Then the map  $LPic A \rightarrow LPic B$  is surjective.*

*Proof.* Since  $A \subseteq B$  is subintegral, so are  $A[X] \subseteq B[X]$  and  $A[X, X^{-1}] \subseteq B[X, X^{-1}]$ . Then the maps  $Pic A[X] \times Pic A[X^{-1}] \rightarrow Pic B[X] \times Pic B[X^{-1}]$  and  $Pic A[X, X^{-1}] \rightarrow Pic B[X, X^{-1}]$  are surjective by Lemma 2.8(1). Hence we get the result by chasing the following commutative diagram

$$\begin{array}{ccccccc} Pic A[X] \times Pic A[X^{-1}] & \longrightarrow & Pic A[X, X^{-1}] & \longrightarrow & LPic A & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \\ Pic B[X] \times Pic B[X^{-1}] & \longrightarrow & Pic B[X, X^{-1}] & \longrightarrow & LPic B & \longrightarrow & 1 \\ \downarrow & & \downarrow & & & & \\ 1 & & 1 & & & & \end{array}$$

□

**Theorem 3.7.** *Let  $A \subseteq B$  be a ring extension with  $A$  hensel local and  $B$  seminormal. Then  $M\mathcal{I}(A, B) \cong M\mathcal{I}(A, {}^+A) \oplus M\mathcal{I}({}^+A, B)$  where  ${}^+A$  is the subintegral closure of  $A$  in  $B$ .*

*Proof.* By Lemma 3.6,  $LPic A \rightarrow LPic {}^+A$  is surjective. Since  $A$  is hensel local,  $LPic A = 0$  by Theorem 2.5 of [8]. Therefore  $LPic {}^+A = 0$  and  $MPic {}^+A = 0$  because  ${}^+A$  is seminormal. Then by same argument as Proposition 3.5(3),  $M\mathcal{I}({}^+A, B)$  is a free abelian group. Now the result follows from the exact sequence

$$1 \rightarrow M\mathcal{I}(A, {}^+A) \rightarrow M\mathcal{I}(A, B) \rightarrow M\mathcal{I}({}^+A, B) \rightarrow 1$$

□

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# SUBINTEGRALITY, INVERTIBLE MODULES AND LAURENT POLYNOMIAL EXTENSIONS

VIVEK SADHU

**ABSTRACT.** Let  $A \subseteq B$  be a commutative ring extension. Let  $\mathcal{I}(A, B)$  be the multiplicative group of invertible  $A$ -submodules of  $B$ . In this article, we extend a result of Sadhu and Singh by finding a necessary and sufficient condition on an integral birational extension  $A \subseteq B$  of integral domains with  $\dim A \leq 1$ , so that the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism. In the same situation, we show that if  $\dim A \geq 2$  then the condition is necessary but not sufficient. We also discuss some properties of the cokernel of the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  in the general case.

**Keywords:** Subintegral extensions, Seminormal rings, Invertible modules

**2010 Mathematics Subject Classification:** 13B02, 13F45

## INTRODUCTION

In [4], Roberts and Singh have introduced the group  $\mathcal{I}(A, B)$  to generalize a result of Dayton. The relation between the group  $\mathcal{I}(A, B)$  and subintegral extensions has been investigated by Reid, Roberts and Singh in a series of papers. Recently in [5], Sadhu and Singh have proved that  $A$  is subintegrally closed in  $B$  if and only if the canonical map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X], B[X])$  is an isomorphism. It is easy to see that the map is injective and that  $\mathcal{I}(A[X], B[X]) = \mathcal{I}(A, B) \oplus N\mathcal{I}(A, B)$ , where  $N\mathcal{I}(A, B)$  denotes the kernel of the map  $\mathcal{I}(A[X], B[X]) \xrightarrow{X \mapsto 0} \mathcal{I}(A, B)$ . So the result of [5], just mentioned, amounts to saying that  $N\mathcal{I}(A, B) = 0$  if and only if  $A$  is subintegrally closed in  $B$ .

The primary goal of this paper is to extend the result of Sadhu and Singh in [5] just mentioned above by finding a necessary and sufficient condition on  $A \subseteq B$ , so that the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism. It is easy to see that the map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is always injective (see Lemma 2.8). Thus the problem reduces to the investigation of conditions for the cokernel of the above map to be zero. This cokernel will be denoted by  $M\mathcal{I}(A, B)$ . The secondary goal will be to investigate properties of the cokernel  $M\mathcal{I}(A, B)$  in the general case.

In Section 1, we mainly give basic definitions and notations.

In Section 2, we discuss conditions on  $A \subseteq B$  under which the map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism. We are able to prove some results in the

situation when  $A \subseteq B$  is an integral birational extension of domains. First, if  $\dim A \leq 1$  then by using a result of Onoda-Yoshida ([3], Theorem 1.13), we prove the following

**Theorem 2.14.** *Let  $A \subseteq B$  be an integral, birational extension of domains with  $\dim A \leq 1$ . Then  $MI(A, B) = 0$  if and only if  $A$  is subintegrally closed in  $B$  and  $A \subseteq B$  is anodal.*

For higher dimension, we show that the above conditions are necessary but not sufficient. More precisely, we prove the following

**Theorem 2.17.** *Let  $A \subseteq B$  be an integral, birational extension of domains. Suppose  $MI(A, B) = 0$ . Then  $A$  is subintegrally closed in  $B$  and  $A \subseteq B$  is anodal.*

That the conditions are not sufficient is shown by an example of C. Weibel (see Remark 2.18). We note that for any ring extension  $A \subseteq B$ , the condition  $MI(A, B) = 0$  implies easily that  $A$  is subintegrally closed in  $B$  (see Lemma 2.15(4)).

In Section 3, we examine the cokernel  $MI(A, B)$  in the general case. In order to do this, we first discuss the surjectivity of the natural map  $\varphi(A, C, B) : \mathcal{I}(A, B) \rightarrow \mathcal{I}(C, B)$  given by  $\varphi(A, C, B)(I) = IC$  for any ring extensions  $A \subseteq C \subseteq B$ . We show that the map  $\varphi(A, C, B)$  is surjective in two cases: (1)  $C$  is subintegral over  $A$ , (2)  $A \subseteq B$  is an integral extension with  $A$  Hensel local (see Propositions 3.1 and 3.2). We show further that if  $C$  is subintegral over  $A$ , then the sequence

$$1 \rightarrow MI(A, C) \rightarrow MI(A, B) \rightarrow MI(C, B) \rightarrow 1$$

is exact (see Proposition 3.3). Finally we prove the following

**Theorem 3.7.** *Let  $A \subseteq B$  be a ring extension with  $A$  Hensel local and  $B$  seminormal. Then  $MI(A, B) \cong MI(A, {}^+A) \oplus MI({}^+A, B)$ , where  ${}^+A$  is the subintegral closure of  $A$  in  $B$ .*

In this section we also observe that if  $A$  is subintegrally closed in  $B$  with  $B$  a seminormal domain and  $A$  Hensel local then  $MI(A, B) = 0$  (see Proposition 3.5(4)).

## 1. BASIC DEFINITIONS AND NOTATIONS

All the rings we consider are commutative with 1, and all ring homomorphisms are unitary. Let  $X, T$  be indeterminates.

An **elementary subintegral** extension is an extension of the form  $A \subseteq B$  with  $B = A[b]$  for some  $b \in B$  such that  $b^2, b^3 \in A$ . An extension  $A \subseteq B$  is **subintegral** if it is a filtered union of elementary subintegral extensions; that is, for each  $b \in B$  there is a finite sequence  $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_r \subseteq B$  of ring extensions such that  $b \in C_r$  and  $C_{i-1} \subseteq C_i$  is elementary subintegral for each  $i$ ,  $1 \leq i \leq r$ . We say that  $A$  is **subintegrally closed** in  $B$  if whenever  $b \in B$  and  $b^2, b^3 \in A$  then  $b \in A$ . The ring  $A$

is **seminormal** if the following condition holds:  $b, c \in A$  and  $b^3 = c^2$  imply that there exists  $a \in A$  with  $b = a^2$  and  $c = a^3$ . A seminormal ring is necessarily reduced and is subintegrally closed in every reduced overring. It is easily seen that if  $A$  is subintegrally closed in  $B$  with  $B$  seminormal then  $A$  is seminormal. For details see [7, 8].

For a ring  $A$  we denote by:

$U(A)$ : The groups of units of  $A$ .

$H^0(A) = H^0(\text{Spec}(A), \mathbb{Z})$ : The group of continuous maps from  $\text{Spec}(A)$  to  $\mathbb{Z}$ .

$\text{Pic } A$ : The Picard group of  $A$ .

$KU(A)$ : Cokernel of the natural map  $U(A) \rightarrow U(A[X])$ .

$MU(A)$ : Cokernel of the natural map  $U(A) \rightarrow U(A[X, X^{-1}])$ .

$NU(A)$ : Kernel of the map  $U(A[X]) \rightarrow U(A)$ .

$\text{KPic } A$ : Cokernel of the natural map  $\text{Pic } A \rightarrow \text{Pic } A[X]$ .

$\text{MPic } A$ : Cokernel of the natural map  $\text{Pic } A \rightarrow \text{Pic } A[X, X^{-1}]$ .

$\text{NPic } A$ : Kernel of the map  $\text{Pic } A[X] \rightarrow \text{Pic } A$ .

$\text{LPic } A$ : Cokernel of the map  $\text{Pic } A[X] \times \text{Pic } A[X^{-1}] \xrightarrow{\text{add}} \text{Pic } A[X, X^{-1}]$ .

Let  $A \subseteq B$  be a ring extension. Then we denote by

$\mathcal{I}(A, B)$ : The group of invertible  $A$ -submodules of  $B$ .

It is easily seen that  $\mathcal{I}$  is a functor from extensions of rings to abelian groups. Some properties of  $\mathcal{I}(A, B)$  can be found in [4, Section 2].

$K\mathcal{I}(A, B)$ : Cokernel of the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X], B[X])$ .

$M\mathcal{I}(A, B)$ : Cokernel of the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ .

$N\mathcal{I}(A, B)$ : Kernel of the map  $\mathcal{I}(A[X], B[X]) \rightarrow \mathcal{I}(A, B)$  (Here the map is induced by the  $B$ -algebra homomorphism  $B[X] \rightarrow B$  given by  $X \mapsto 0$ ).

Recall from [4, Section 2] that for any commutative ring extension  $A \subseteq B$ , we have the exact sequence

$$1 \rightarrow U(A) \rightarrow U(B) \rightarrow \mathcal{I}(A, B) \rightarrow \text{Pic } A \rightarrow \text{Pic } B.$$

Applying  $M, K$  we obtain the chain complexes:

$$(1.0) \quad 1 \rightarrow MU(A) \rightarrow MU(B) \rightarrow M\mathcal{I}(A, B) \xrightarrow{\eta} \text{MPic } A \xrightarrow{\varphi} \text{MPic } B$$

and

$$(1.1) \quad 1 \rightarrow KU(A) \rightarrow KU(B) \rightarrow K\mathcal{I}(A, B) \xrightarrow{\alpha} \text{KPic } A \xrightarrow{\beta} \text{KPic } B.$$

## 2. THE MAP $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$

In this section we examine some conditions on  $A \subseteq B$  under which the natural map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$  is an isomorphism. For this we consider the notions of quasinormal and anodal extensions (or  $u$ -closed).

Let  $A \subseteq B$  be a ring extension. We say that  $A$  is **quasinormal** in  $B$  if the natural map  $MPic A \rightarrow MPic B$  is injective. For properties of quasinormal extensions see [3].

An inclusion  $A \subseteq B$  of rings is called **anodal** or **an anodal extension**, if every  $b \in B$  such that  $(b^2 - b) \in A$  and  $(b^3 - b^2) \in A$  belongs to  $A$ . This notion was first introduced by Asanuma and Onoda-Yoshida in [3], and they called this notion ' $u$ -closed'. Some related details can be found in [1, 3, 9].

We first show in Proposition 2.2 below that a subintegral extension is always an anodal extension, which is perhaps a result of independent interest.

**Lemma 2.1.** *Let  $A \subseteq C \subseteq B$  be extensions of rings. Then the following statements hold:*

- (1) *If  $A$  is anodal in  $B$ , then so is  $A$  in  $C$ .*
- (2) *If  $A$  is anodal in  $C$  and  $C$  is anodal in  $B$ , then so is  $A$  in  $B$ .*

*Proof.* Clear from the definition. □

**Proposition 2.2.** *Let  $A \subseteq B$  be a ring extension. If  $A \subseteq B$  is subintegral, then it is anodal.*

*Proof.* Assume first that  $A \subseteq B$  is an elementary subintegral extension, i.e.,  $A \subseteq B = A[b]$  such that  $b^2, b^3 \in A$ . Let  $f \in B$  such that  $f^2 - f, f^3 - f^2 \in A$ . We have to show that  $f \in A$ . Clearly  $f$  is of the form  $a + \lambda b$  where  $a, \lambda \in A$ . So it is enough to show that  $\lambda b \in A$ . Since  $\lambda b(2a - 1), \lambda b(3a^2 - 1) \in A$ ,  $\lambda b = \lambda b \cdot 1 = \lambda b[(6a + 3)(2a - 1) - 4(3a^2 - 1)] \in A$ . Hence  $f \in A$ .

In the general case, for  $f \in B$  there exists a finite sequence  $A = C_0 \subseteq C_1 \subseteq \dots \subseteq C_r \subseteq B$  of extensions such that  $C_i \subseteq C_{i+1}$  is an elementary subintegral extension for each  $i, 0 \leq i \leq r - 1$  and  $f \in C_r$ . So by the above argument  $C_i \subseteq C_{i+1}$  is anodal for each  $i$ . Now the result follows from Lemma 2.1(2). □

The following result is due to Sadhu and Singh ([5], Theorem 1.5) which we use frequently throughout this paper:

**Lemma 2.3.** *Let  $A \subseteq B$  be a ring extension. Then  $A$  is subintegrally closed in  $B$  if and only if the canonical map  $\mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X], B[X])$  is an isomorphism.* □

One can restate the above result in the following way:  $A$  is subintegrally closed in  $B \Leftrightarrow KI(A, B) = 0 \Leftrightarrow NI(A, B) = 0$ .



The following result is due to Weibel ([9], Theorem 5.2).

**Lemma 2.4.** *There is a natural decomposition*

$$\mathrm{Pic} A[X, X^{-1}] \cong \mathrm{Pic} A \oplus \mathrm{NPic} A \oplus \mathrm{NPic} A \oplus \mathrm{LPic} A$$

for any commutative ring  $A$ . □

**Remark 2.5.** By Swan Theorem [7],  $\mathrm{NPic} A = 0$  if and only if  $A_{red}$  is seminormal. So for a seminormal ring  $A$ ,  $\mathrm{LPic} A \cong \mathrm{MPic} A$ .

The next result is given in ([10], Exercise 3.17, Page 30).

**Lemma 2.6.** *There is a natural decomposition*

$$U(A[X, X^{-1}]) \cong U(A) \oplus \mathrm{NU}(A) \oplus \mathrm{NU}(A) \oplus H^0(A)$$

for any commutative ring  $A$ . □

**Remark 2.7.** It follows that for a reduced ring  $A$ ,  $H^0(A) \cong \mathrm{MU}(A)$ .

**Lemma 2.8.** *The natural map  $\phi : \mathcal{I}(A, B) \rightarrow \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}])$ , given by  $I \rightarrow IA[X, X^{-1}]$ , is injective. Thus,  $\phi$  is an isomorphism if and only if  $M\mathcal{I}(A, B) = 0$ .*

*Proof.* Let  $I = (b_1, b_2, \dots, b_r)A \in \ker \phi$ , where  $b_i \in B$ . Then  $IA[X, X^{-1}] = A[X, X^{-1}]$ . This implies that  $b_i \in A[X, X^{-1}] \cap B = A$ , for all  $i$ . So  $I \subseteq A$ . Similarly  $I^{-1} \subseteq A$ . Hence  $I = A$ . □

**Lemma 2.9.** *The sequence (1.0)[respectively (1.1)] is exact, except possibly at the place  $\mathrm{MPic} A$  [respectively  $K\mathrm{Pic} A$ ]. It is exact there too if the map  $\mathrm{Pic} A \rightarrow \mathrm{Pic} B$  is surjective.*

*Proof.* We have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & U(A) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \mathrm{Pic} A & \longrightarrow & \mathrm{Pic} B \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & U(A[X, X^{-1}]) & \longrightarrow & U(B[X, X^{-1}]) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & \mathrm{Pic} A[X, X^{-1}] & \longrightarrow & \mathrm{Pic} B[X, X^{-1}] \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathrm{MU}(A) & \longrightarrow & \mathrm{MU}(B) & \longrightarrow & M\mathcal{I}(A, B) & \longrightarrow & \mathrm{MPic} A & \longrightarrow & \mathrm{MPic} B \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1 & & 0 & & 0
 \end{array}$$

where the first two rows are exact and each column is exact. Now the result follows by chasing this diagram.  $\square$

**Lemma 2.10.** *Let  $A \subseteq B$  be a ring extension. The map  $\text{Pic } A \rightarrow \text{Pic } B$  is surjective if any one of the following conditions holds:*

- (1)  $A \subseteq B$  is subintegral.
- (2)  $A \subseteq B$  is an integral, birational extension of domains with  $\dim A \leq 1$ .

*Proof.* (1) See Proposition 7 of [2].

(2) Let  $K$  be the quotient field of  $A$  and  $B$ . We have the commutative diagram

$$\begin{array}{ccccc} \mathcal{I}(A, K) & \longrightarrow & \text{Pic } A & \longrightarrow & 0 \\ \varphi(A, B, K) \downarrow & & \rho \downarrow & & \\ \mathcal{I}(B, K) & \longrightarrow & \text{Pic } B & \longrightarrow & 0 \end{array}$$

where  $\varphi(A, B, K)$  is surjective by Proposition 2.3 of [5]. Hence  $\rho$  is surjective.  $\square$

**Lemma 2.11.** (cf. [3], Lemma 1.4.) *Let  $A \subseteq B$  be a ring extension with  $B$  reduced and  $A$  quasinormal in  $B$ . Then  $A$  is subintegrally closed in  $B$ .*

*Proof.* We have not assumed  $B$  to be a domain. By Lemma 2.3, it is enough to show that  $K\mathcal{I}(A, B) = 0$ . We have the sequence

$$1 \rightarrow KU(A) \rightarrow KU(B) \rightarrow K\mathcal{I}(A, B) \xrightarrow{\alpha} KPic A \xrightarrow{\beta} KPic B$$

which is exact except possibly at the place  $KPic A$ . Since  $A$  and  $B$  are reduced,  $KU(A) = 0$  and  $KU(B) = 0$ . In the proof of Lemma 1.4 of [3], it is shown that the map  $KPic A \rightarrow KPic B$  is injective, i.e.,  $\ker \beta = 0$ . We have  $\text{im } \alpha \subseteq \ker \beta$ . Hence  $K\mathcal{I}(A, B) = 0$ .  $\square$

**Remark 2.12.** In the above lemma we cannot drop the condition that  $B$  is reduced. For example, consider the extension  $A = K \subsetneq B = K[b]$  with  $b^2 = 0$ , where  $K$  is any field. Since  $MPic K = 0$ , clearly  $A$  is quasinormal in  $B$ . But  $A$  is not subintegrally closed in  $B$ , because  $b^2 = b^3 = 0 \in K$ ,  $b \notin K$ .

**Lemma 2.13.** *Let  $A \subseteq B$  be a ring extension with  $B$  a domain. Then the following statements hold:*

- (1) If  $A$  is quasinormal in  $B$  then  $M\mathcal{I}(A, B) = 0$ .
- (2) Suppose the extension  $A \subseteq B$  is integral and birational with  $\dim A \leq 1$ , and  $M\mathcal{I}(A, B) = 0$ . Then  $A$  is quasinormal in  $B$ .

*Proof.* (1) Since  $A$  and  $B$  are domains,  $MU(A) = MU(B) \cong \mathbb{Z}$ . By (1.0),  $\text{im } \eta \subseteq \ker \varphi$ . As  $A$  is quasinormal in  $B$ ,  $\ker \varphi = 0$ . This implies that  $\text{im } \eta = 0$ . We get  $M\mathcal{I}(A, B) = 0$ .

(2) By Lemma 2.10(2) and Lemma 2.9, the sequence (1.0) is exact at  $MPic A$  also. Since  $M\mathcal{I}(A, B) = 0$ , we get the result.  $\square$

**Theorem 2.14.** *Let  $A \subseteq B$  be an integral, birational extension of domains with  $\dim A \leq 1$ . Then  $M\mathcal{I}(A, B) = 0$  if and only if  $A$  is subintegrally closed in  $B$  and  $A \subseteq B$  is anodal.*

*Proof.* If  $\dim A = 0$  then  $A = B$  and the assertion holds trivially in this case. If  $\dim A = 1$  then by Theorem 1.13 of [3],  $A$  is quasinormal in  $B$  if and only if  $A$  is subintegrally closed in  $B$  and  $A \subseteq B$  is anodal. We also have  $A$  is quasinormal in  $B$  if and only if  $M\mathcal{I}(A, B) = 0$  by Lemma 2.13. Combining these two results we get the assertion.  $\square$

Next, in Theorem 2.17 and Remark 2.18, we show that in general, the conditions  $A$  is subintegrally closed in  $B$  and  $A \subseteq B$  is anodal are necessary but not sufficient.

**Lemma 2.15.** (1) *The diagram*

$$\begin{array}{ccc} \mathcal{I}(A, B) & \xrightarrow{\psi} & \mathcal{I}(A[X], B[X]) \\ & \searrow \phi & \downarrow \theta \\ & & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \end{array}$$

*is commutative.*

(2) *The maps  $\phi$ ,  $\psi$  and  $\theta$  are injective.*

(3)  *$\phi$  is an isomorphism if and only if  $\psi$  and  $\theta$  are isomorphisms.*

(4) *If  $\phi$  is an isomorphism, i.e.,  $M\mathcal{I}(A, B) = 0$ , then  $A$  is subintegrally closed in  $B$ .*

*Proof.* (1) Since the maps are natural, the diagram is commutative.

(2)  $\phi$  is injective by Lemma 2.8. The injectivity of  $\psi$  and  $\theta$  follows by a similar argument as in Lemma 2.8.

(3) If  $\psi$  and  $\theta$  are isomorphisms then clearly  $\phi$  is an isomorphism. Conversely, suppose  $\phi$  is an isomorphism. Then by simple diagram chasing we get that  $\psi$  and  $\theta$  are isomorphisms.

(4) If  $\phi$  is an isomorphism then  $\psi$  is an isomorphism. Hence by Lemma 2.3,  $A$  is subintegrally closed in  $B$ .  $\square$

**Lemma 2.16.** *Let  $\mathfrak{a}$  be a  $B$ -ideal contained in  $A$ . Then the homomorphism  $M\mathcal{I}(A, B) \rightarrow M\mathcal{I}(A/\mathfrak{a}, B/\mathfrak{a})$  is an isomorphism.*

*Proof.* Clearly,  $\mathfrak{a}[X, X^{-1}]$  is a  $B[X, X^{-1}]$ -ideal contained in  $A[X, X^{-1}]$ . We have  $\mathcal{I}(A, B) \cong \mathcal{I}(A/\mathfrak{a}, B/\mathfrak{a})$  and  $\mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) \cong \mathcal{I}(A/\mathfrak{a}[X, X^{-1}], B/\mathfrak{a}[X, X^{-1}])$  by Proposition 2.6 of [4]. Now by chasing a suitable diagram we get the result.  $\square$

**Theorem 2.17.** *Let  $A \subseteq B$  be an integral, birational extension of domains. Suppose  $MI(A, B) = 0$ . Then  $A$  is subintegrally closed in  $B$  and  $A \subseteq B$  is anodal.*

*Proof.* By Lemma 2.15(4),  $A$  is subintegrally closed in  $B$ . To prove  $A \subseteq B$  is anodal, by Lemma 1.10 of [3], it is enough to show that for every intermediate ring  $C$  between  $A$  and  $B$  such that  $C$  is a finite  $A$ -module, the map  $MPic A \rightarrow MPic(A/\mathfrak{c}) \times MPic C$  is injective, where  $\mathfrak{c}$  is the conductor of  $C$  in  $A$ . We first claim that the map  $\tau : MI(A, C) \rightarrow MI(A, B)$  is injective, where  $C$  is any intermediate ring between  $A$  and  $B$ .

We have the commutative diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathcal{I}(A, C) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}]) & \longrightarrow & MI(A, C) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \tau \\ 1 & \longrightarrow & \mathcal{I}(A, B) & \xrightarrow{\phi} & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & MI(A, B) \longrightarrow 1 \end{array}$$

where the first two vertical arrows are natural inclusions (because any invertible  $A$ -submodule of  $C$  is also an invertible  $A$ -submodule of  $B$ ).

Let  $\bar{J} \in \ker \tau$ , where  $J \in \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}])$ . Then  $J \in \text{im } \phi$  and there exists  $J_1 \in \mathcal{I}(A, B)$  such that  $J_1 A[X, X^{-1}] = J$ . Let  $J_1 = (b_1, b_2, \dots, b_r)A$  and  $J = (f_1, f_2, \dots, f_s)A[X, X^{-1}]$  where  $b_i \in B$  and  $f_i \in C[X, X^{-1}]$ . Then clearly  $b_i \in B \cap C[X, X^{-1}] = C$  for all  $i$ . So  $J_1 \subseteq C$ . Also  $J_1^{-1} \subseteq C$ . This implies that  $J_1 \in \mathcal{I}(A, C)$ . So  $\bar{J} = 0$ . This proves the claim.

Since  $MI(A, B) = 0$ ,  $MI(A, C) = 0$ . By Lemma 2.16,  $MI(A/\mathfrak{c}, C/\mathfrak{c}) = 0$ , where  $\mathfrak{c}$  is the conductor of  $C$  in  $A$ . By (1.0), we have  $MU(A) \cong MU(C)$  and  $MU(A/\mathfrak{c}) \cong MU(C/\mathfrak{c})$ . Now the result follows from the following exact sequence which we obtain by applying  $M$  to the unit-Pic sequence ([10], Theorem 3.10),

$$MU(A) \rightarrow MU(A/\mathfrak{c}) \times MU(C) \rightarrow MU(C/\mathfrak{c}) \rightarrow MPic A \rightarrow MPic(A/\mathfrak{c}) \times MPic C$$

$\square$

**Remark 2.18.** The converse of the above theorem holds for  $\dim A \leq 1$  as seen in Theorem 2.14. In general, the converse does not hold. This is seen by considering Example 3.5 of C. Weibel [9]. In that example  $A$  is a 2-dimensional noetherian domain whose integral closure is  $B = K[X, Y]$ , where  $K$  is a field. So  $A \subseteq B$  is an integral, birational extension. By Proposition 3.5.2 of [9],  $A \subseteq B$  is anodal and  $A$  is subintegrally

closed in  $B$ . Since  $B$  is a UFD,  $\text{Pic } B = \text{Pic } B[T, T^{-1}] = 0$ . Then we get the exact sequence

$$1 \rightarrow MU(A) \rightarrow MU(B) \rightarrow M\mathcal{I}(A, B) \rightarrow MPic A \rightarrow 0.$$

As  $A, B$  are domains,  $MU(A) = MU(B) \cong \mathbb{Z}$ . So  $M\mathcal{I}(A, B) \cong MPic A$ . By Remark 2.5,  $LPic A \cong MPic A$ . Hence by Proposition 3.5.2 of [9],  $M\mathcal{I}(A, B) \neq 0$ .

### 3. SOME OBSERVATIONS ON $M\mathcal{I}(A, B)$

In this section we discuss some properties of the cokernel  $M\mathcal{I}(A, B)$  in the general case.

Recall from [6, Section 3] that for any extensions  $A \subseteq C \subseteq B$  of rings, we have the exact sequence

$$1 \rightarrow \mathcal{I}(A, C) \rightarrow \mathcal{I}(A, B) \xrightarrow{\varphi(A, C, B)} \mathcal{I}(C, B)$$

where the map  $\varphi(A, C, B)$  is given by  $\varphi(A, C, B)(I) = IC$ .

Now it is natural to ask under what conditions on  $A \subseteq B$  the map  $\varphi(A, C, B)$  is surjective. In [6], Singh has proved that if  $B$  is subintegral over  $A$  then the map  $\varphi(A, C, B)$  is surjective. In the next Proposition we generalize Singh's result as follows:

**Proposition 3.1.** *For all rings  $C$  between  $A$  and  $B$  such that  $C$  is subintegral over  $A$ , the map  $\varphi(A, C, B)$  is surjective.*

*Proof.* We have the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(A) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \text{Pic } A & \longrightarrow & \text{Pic } B \\ & & \downarrow & & \downarrow = & & \downarrow \varphi(A, C, B) & & \downarrow \rho & & \downarrow = \\ 1 & \longrightarrow & U(C) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(C, B) & \longrightarrow & \text{Pic } C & \longrightarrow & \text{Pic } B \end{array}$$

Since  $\rho$  is surjective by Lemma 2.10(1), the result follows by chasing the diagram.  $\square$

The following result gives another case where the map  $\varphi(A, C, B)$  is surjective.

Recall that a local ring  $A$  is **Hensel** if every finite  $A$ -algebra  $B$  is a direct product of local rings.

**Proposition 3.2.** *Let  $A \subseteq B$  be an integral extension with  $A$  Hensel local. Then for all rings  $C$  with  $A \subseteq C \subseteq B$  the map  $\varphi(A, C, B)$  is surjective.*

*Proof.* By Lemma 2.2 of [5], it is enough to show that  $\varphi(A, D, B)$  is surjective for every subring  $D$  of  $C$  containing  $A$  such that  $D$  is finitely generated as an  $A$ -algebra. Let such a ring  $D$  be given. Since  $D$  is integral over  $A$ ,  $D$  is a finite  $A$ -algebra. As  $A$  is Hensel,  $D$  is a finite direct product of local rings. Then  $\text{Pic } A$  and  $\text{Pic } D$  are both trivial. This

implies that  $\mathcal{I}(A, B) = U(B)/U(A)$ ,  $\mathcal{I}(D, B) = U(B)/U(D)$  and clearly  $\varphi(A, D, B)$  is surjective.  $\square$

**Proposition 3.3.** *Let  $A \subseteq C \subseteq B$  be extensions of rings with  $A \subseteq C$  subintegral. Then the sequence*

$$1 \rightarrow M\mathcal{I}(A, C) \rightarrow M\mathcal{I}(A, B) \rightarrow M\mathcal{I}(C, B) \rightarrow 1$$

*is exact.*

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc}
 & & 1 & & 1 & & 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathcal{I}(A, C) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], C[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, C) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, B) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 1 & \longrightarrow & \mathcal{I}(C, B) & \longrightarrow & \mathcal{I}(C[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(C, B) \longrightarrow 1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 1 & & 1 & & 1
 \end{array}$$

where the rows are clearly exact. Since  $A \subseteq C$  is subintegral, so is  $A[X, X^{-1}] \subseteq C[X, X^{-1}]$ . Therefore by Proposition 3.1, the first two columns are exact. Hence exactness of the last column follows by chasing the diagram.  $\square$

**Corollary 3.4.** *Let  $A \subseteq B$  be a ring extension and let  ${}^+A$  denote the subintegral closure of  $A$  in  $B$ . Then the sequence*

$$1 \rightarrow M\mathcal{I}(A, {}^+A) \rightarrow M\mathcal{I}(A, B) \rightarrow M\mathcal{I}({}^+A, B) \rightarrow 1$$

*is exact.*

*Proof.* Immediate from Proposition 3.3.  $\square$

**Proposition 3.5.** *Let  $A \subseteq B$  be a ring extension. Assume that  $A$  is subintegrally closed in  $B$ . Then*

- (1)  $M\mathcal{I}(A, B) \cong M\mathcal{I}(A[T], B[T])$ .
- (2)  $M\mathcal{I}(A, B)$  is a torsion-free abelian group if  $B$  is a seminormal ring.
- (3)  $M\mathcal{I}(A, B)$  is a free abelian group if  $B$  is a seminormal ring and  $A$  is Hensel local.
- (4)  $M\mathcal{I}(A, B) = 0$  if  $B$  is a seminormal domain and  $A$  is Hensel local.

*Proof.* (1) Since  $A$  is subintegrally closed in  $B$ ,  $A[X]$  is subintegrally closed in  $B[X]$  by Corollary 1.6 of [5]. Therefore  $A[X, X^{-1}]$  is subintegrally closed in  $B[X, X^{-1}]$ . We have the commutative diagram

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \mathcal{I}(A, B) & \longrightarrow & \mathcal{I}(A[X, X^{-1}], B[X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A, B) \longrightarrow 1 \\
 & & \downarrow \psi & & \downarrow \xi & & \downarrow \\
 1 & \longrightarrow & \mathcal{I}(A[T], B[T]) & \longrightarrow & \mathcal{I}(A[T][X, X^{-1}], B[T][X, X^{-1}]) & \longrightarrow & M\mathcal{I}(A[T], B[T]) \longrightarrow 1
 \end{array}$$

where  $\psi$  and  $\xi$  are isomorphisms by Lemma 2.3. Hence we get the result.

(2) As  $A$  is subintegrally closed in  $B$  and  $B$  is a seminormal ring,  $A$  is seminormal. Then by Remark 2.5,  $LPic A \cong MPic A$ . Since any seminormal ring is reduced,  $MU(A) = H^0(A)$  and  $MU(B) = H^0(B)$  by Remark 2.7. Now, from (1.0), we have the exact sequence

$$1 \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow M\mathcal{I}(A, B) \rightarrow MPic A$$

where  $MPic A$  is a torsion-free abelian group by Corollary 2.3.1 of [9]. Let  $T$  be the cokernel of the map  $H^0(A) \rightarrow H^0(B)$ . Then

$$1 \rightarrow T \rightarrow M\mathcal{I}(A, B) \rightarrow MPic A$$

is exact and  $T$  is a free abelian group by Proposition 1.3 of [9]. Therefore  $M\mathcal{I}(A, B)$  is a torsion-free abelian group.

(3) By Theorem 2.5 of [9],  $LPic A = 0$ . Since  $A$  is seminormal,  $MPic A = 0$ . Then we have the exact sequence

$$1 \rightarrow H^0(A) \rightarrow H^0(B) \rightarrow M\mathcal{I}(A, B) \rightarrow 1$$

and  $M\mathcal{I}(A, B) = \text{Coker}[H^0(A) \rightarrow H^0(B)]$  is a free abelian group by Proposition 1.3 of [9].

(4) Since  $B$  is a domain,  $H^0(A) = H^0(B) \cong \mathbb{Z}$ . So  $M\mathcal{I}(A, B) = 0$ . □

**Lemma 3.6.** *Let  $A \subseteq B$  be a subintegral extension. Then the map  $LPic A \rightarrow LPic B$  is surjective.*

*Proof.* Since  $A \subseteq B$  is subintegral, so are  $A[X] \subseteq B[X]$  and  $A[X, X^{-1}] \subseteq B[X, X^{-1}]$ . Then the maps  $\text{Pic } A[X] \times \text{Pic } A[X^{-1}] \rightarrow \text{Pic } B[X] \times \text{Pic } B[X^{-1}]$  and  $\text{Pic } A[X, X^{-1}] \rightarrow \text{Pic } B[X, X^{-1}]$  are surjective by Lemma 2.10(1). Hence we get the result by chasing the

following commutative diagram

$$\begin{array}{ccccccc}
\text{Pic } A[X] \times \text{Pic } A[X^{-1}] & \longrightarrow & \text{Pic } A[X, X^{-1}] & \longrightarrow & \text{LPic } A & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Pic } B[X] \times \text{Pic } B[X^{-1}] & \longrightarrow & \text{Pic } B[X, X^{-1}] & \longrightarrow & \text{LPic } B & \longrightarrow & 1 \\
\downarrow & & \downarrow & & & & \\
1 & & 1 & & & & 
\end{array}$$

□

**Theorem 3.7.** *Let  $A \subseteq B$  be a ring extension with  $A$  Hensel local and  $B$  seminormal. Then  $MI(A, B) \cong MI(A, {}^+A) \oplus MI({}^+A, B)$ , where  ${}^+A$  is the subintegral closure of  $A$  in  $B$ .*

*Proof.* By Lemma 3.6,  $\text{LPic } A \rightarrow \text{LPic } {}^+A$  is surjective. Since  $A$  is Hensel local,  $\text{LPic } A = 0$  by Theorem 2.5 of [9]. Therefore  $\text{LPic } {}^+A = 0$  and  $\text{MPic } {}^+A = 0$  because  ${}^+A$  is seminormal. Then by the same argument as Proposition 3.5(3),  $MI({}^+A, B)$  is a free abelian group. Now the result follows from the following exact sequence (Corollary 3.4)

$$1 \rightarrow MI(A, {}^+A) \rightarrow MI(A, B) \rightarrow MI({}^+A, B) \rightarrow 1$$

□

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